

Stochastic inflation, non-perturbative effects, and primordial black holes

UNIVERSITY OF BARCELONA

16 DECEMBER 2025

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Stochastic inflation

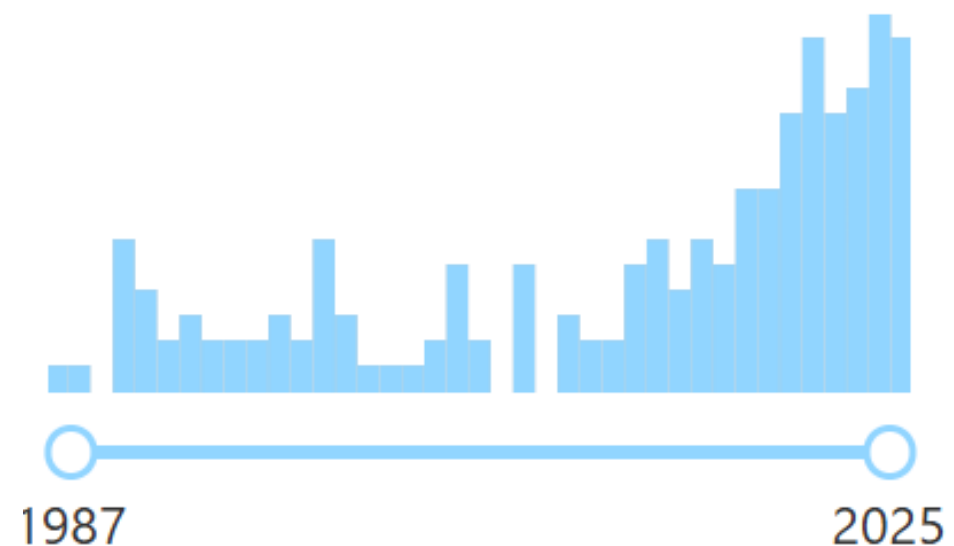
Cosmological perturbations
beyond perturbation theory

Initial paper: Starobinsky
1986

“Stochastic inflation” in
INSPIRE-HEP

- 164 papers
- 98 of these since 2015

Date of paper



Background

Equations of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Slow-roll parameters:

$$\epsilon_1 \equiv -\partial_N \ln H, \quad \epsilon_2 \equiv \partial_N \ln \epsilon_1$$

 Inflation?

 Slow-roll?

Perturbations

Field perturbations:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[\frac{k^2}{a^2} + V''(\phi) - \frac{1}{a^3} \frac{d}{dt} \left(\frac{a^3}{H} \dot{\phi}^2 \right) \right] \delta\phi = 0$$

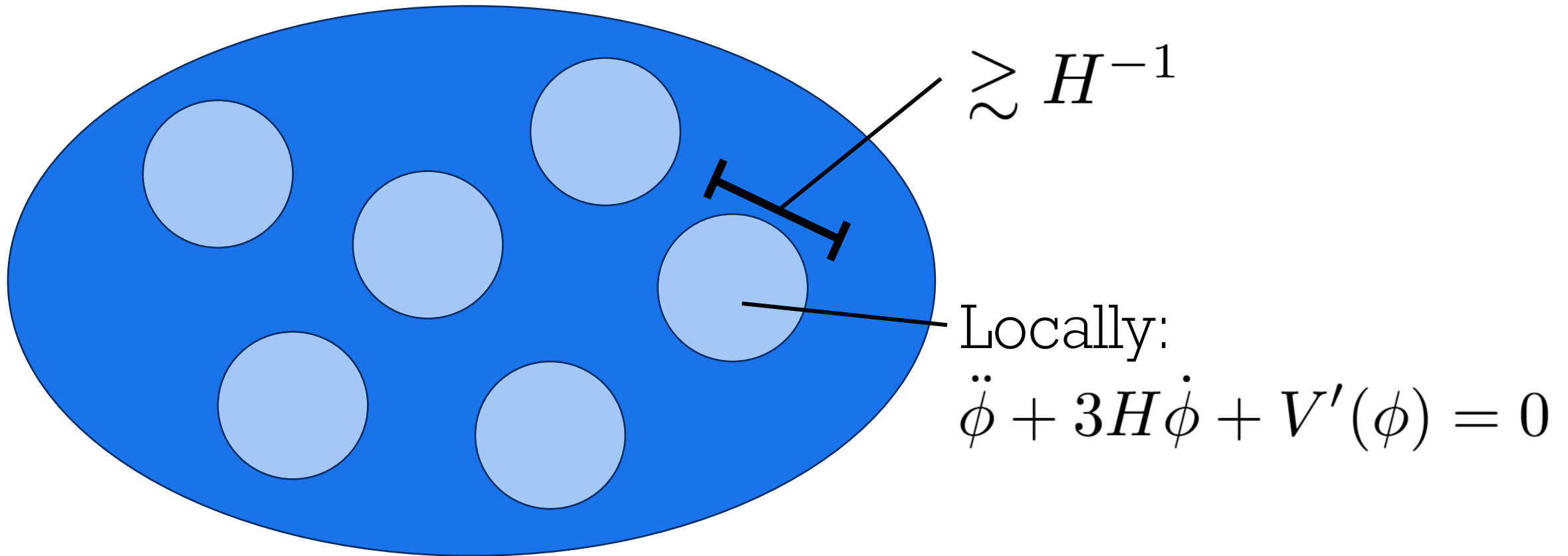
Curvature perturbation:

$$\zeta_k = \frac{\delta\phi_k}{\sqrt{2\epsilon_1}} \leftarrow \sim H$$

$$\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta_k|^2$$

Beyond linear perturbations


Separate universe approximation:




ΔN approximation: $\Delta N = N - \langle N \rangle = \zeta$

Divide the field in two

$$\phi_{\text{tot}} = \phi + \delta\phi$$


$$\int_{k < k_\sigma} \frac{dk^3}{(2\pi)^{2/3}} \phi_k(t) e^{-i\vec{k}\cdot\vec{x}}$$

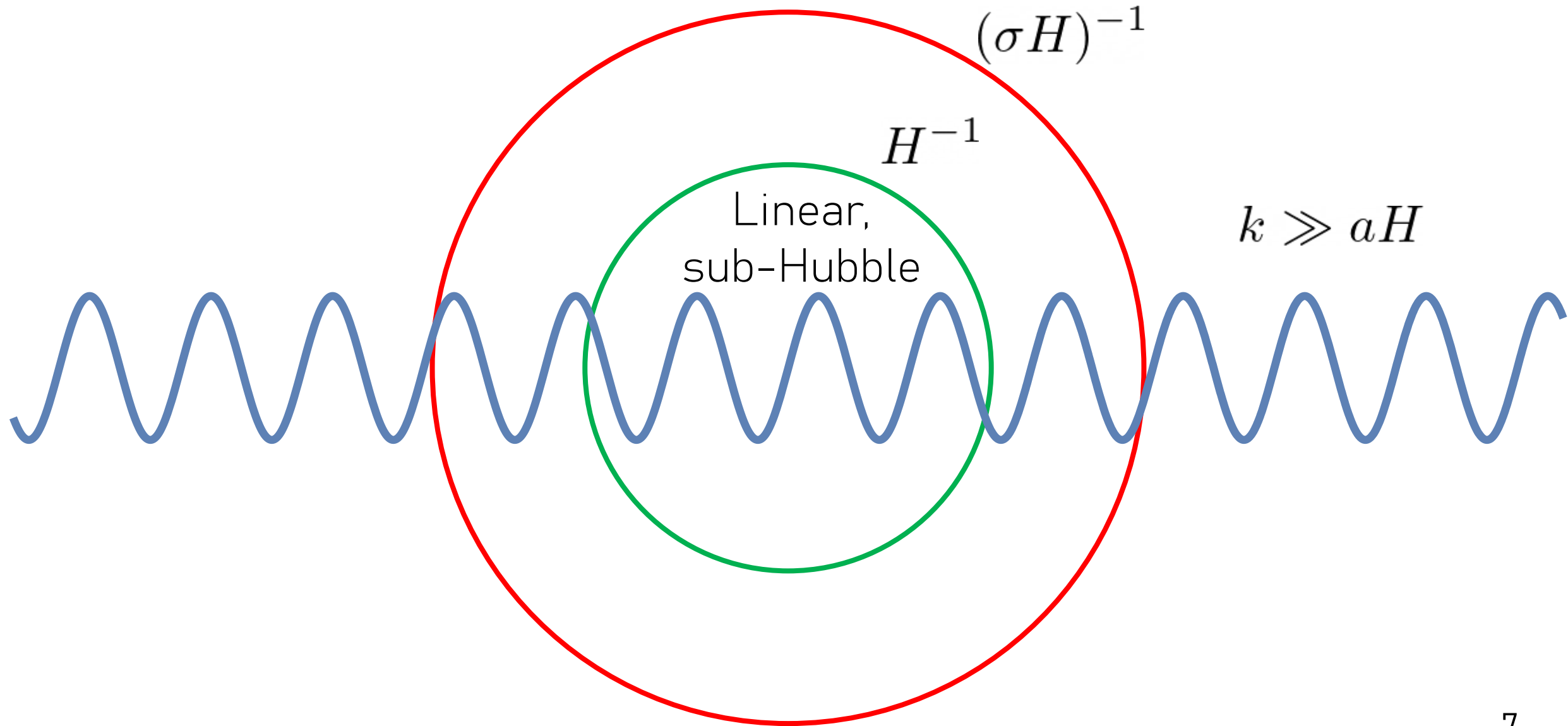
Separate universe


$$\int_{k > k_\sigma} \frac{dk^3}{(2\pi)^{2/3}} \phi_k(t) e^{-i\vec{k}\cdot\vec{x}}$$

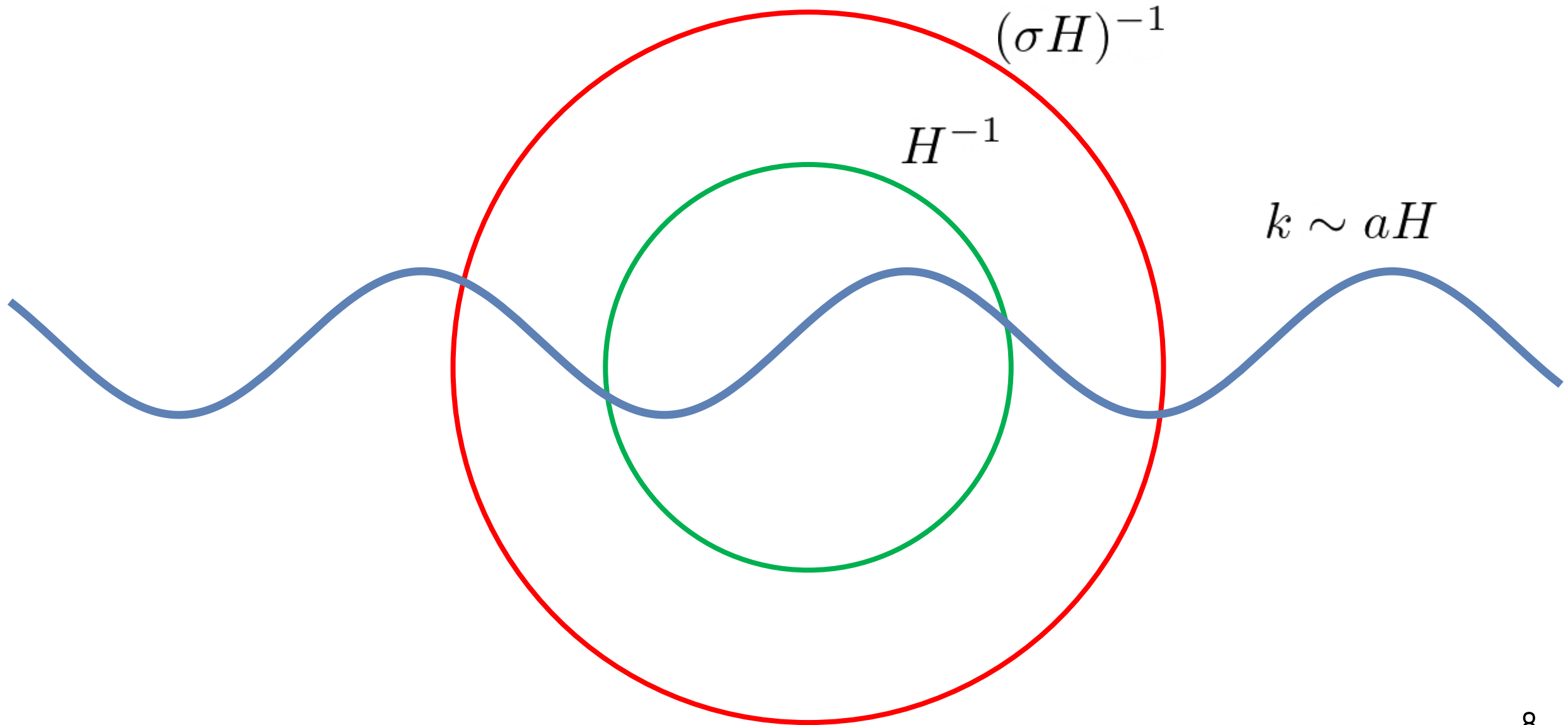
Linear perturbations

Coarse-graining scale: $k = k_\sigma \equiv \sigma aH$

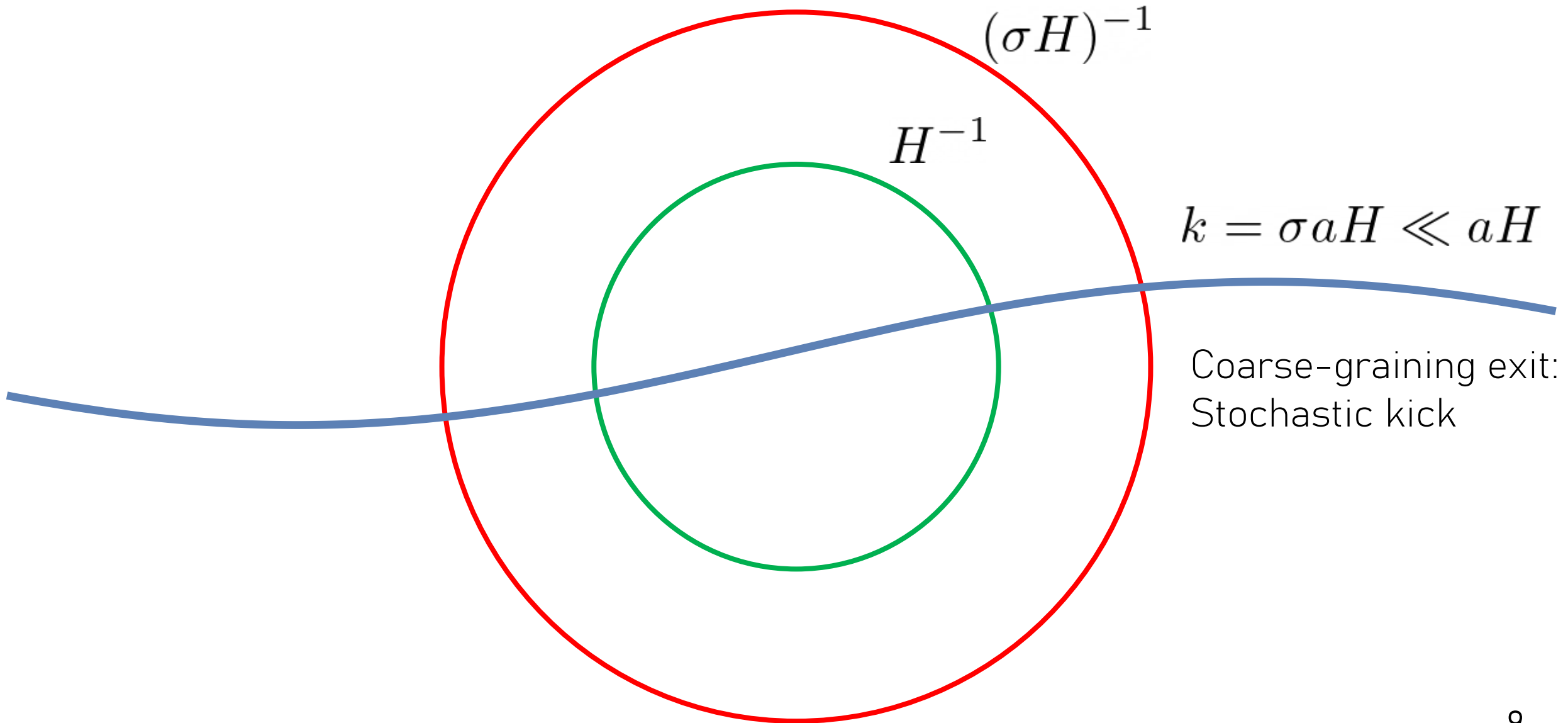
Drifting modes: stochasticity



Drifting modes: stochasticity



Drifting modes: stochasticity



Stochastic inflation

$$\partial_N \phi = \pi + \xi_\phi, \quad \partial_N \pi = - \left(3 - \frac{1}{2} \pi^2 \right) \left(\pi + \frac{V'(\phi)}{V(\phi)} \right) + \xi_\pi$$

e-folds as time variable

$$\langle \xi_X(N) \xi_Y(\tilde{N}) \rangle = \mathcal{P}_{XY}(N, k_\sigma) \delta(N - \tilde{N})$$



de Sitter result / numerical computation

Stochastic inflation

Slow roll:

$$\partial_N \phi = -\frac{V'(\phi)}{V(\phi)} + \xi_\phi$$


$$3H^2(\phi) = V(\phi)$$

$$\langle \xi_\phi(N) \xi_\phi(\tilde{N}) \rangle = \frac{H^2(\phi)}{4\pi^2} \delta(N - \tilde{N})$$

Stochastic inflation


Slow roll:

$$\partial_N \phi = -\frac{V'(\phi)}{V(\phi)} + \xi_\phi$$

 $\mu(\phi)$

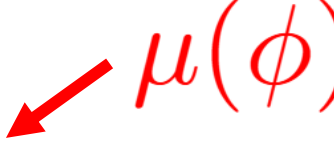
$$\langle \xi_\phi(N) \xi_\phi(\tilde{N}) \rangle = \frac{H^2(\phi)}{4\pi^2} \delta(N - \tilde{N})$$

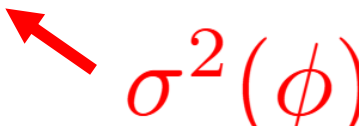
$$\partial_N P(\phi, N) = \partial_\phi \left[\partial_\phi \left(\frac{1}{2} \sigma^2(\phi) P(\phi, N) \right) - \mu(\phi) P(\phi, N) \right]$$

 $\sigma^2(\phi)$

Stochastic inflation

Constant roll:

$$\partial_N \phi = \frac{\epsilon_2}{2} \phi + \xi_\phi$$


$$\langle \xi_\phi(N) \xi_\phi(\tilde{N}) \rangle = \frac{H^2}{4\pi^2} \frac{\sigma^3 \pi}{2} |H_\nu(\sigma)|^2 \delta(N - \tilde{N})$$


$$\partial_N P(\phi, N) = \partial_\phi \left[\partial_\phi \left(\frac{1}{2} \sigma^2(\phi) P(\phi, N) \right) - \mu(\phi) P(\phi, N) \right]$$

Initial ϕ



stochastic
evolution



Switch noise off



classical
evolution



Final ϕ

$$N = 0$$

Final coarse-graining scale
 $k = \sigma aH$

final $N = N_{<k}$

Delta N formalism

$$\begin{aligned}\Delta N_{<k} &= N_{<k} - \langle N_{<k} \rangle \\ &= \zeta_{<k}(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^{3/2}} \zeta_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \theta(k - p)\end{aligned}$$

Delta N formalism

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Vary coarse-graining scale for same patch

Delta N formalism

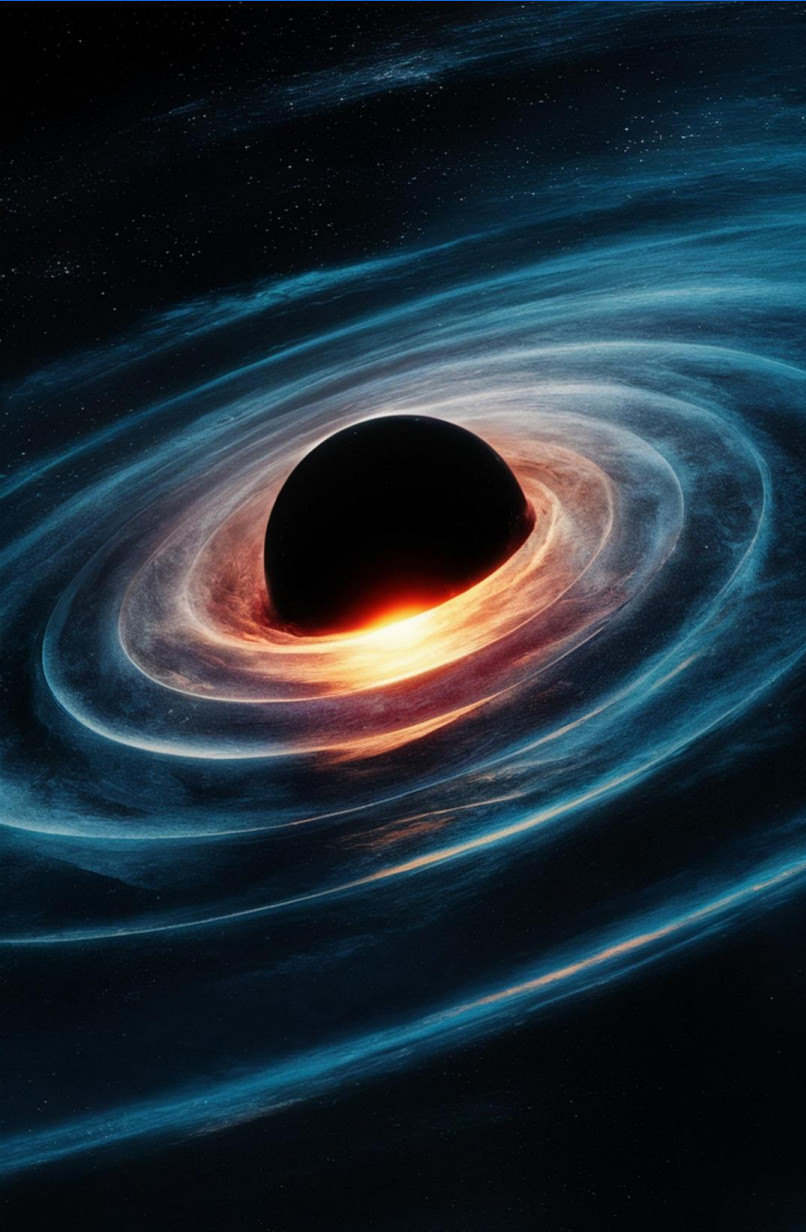
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Vary coarse-graining scale for same patch

$$\text{Obtain: } \zeta_k = \sqrt{\frac{\pi}{2}} \frac{1}{k^3} \frac{d\zeta_{<k}}{d \ln k} \rightarrow \zeta(r) = \int_0^\infty \frac{dk}{k} \frac{d\zeta_{<k}}{d \ln k} \frac{\sin(kr)}{kr}$$

Primordial black holes

High peaks: primordial black holes

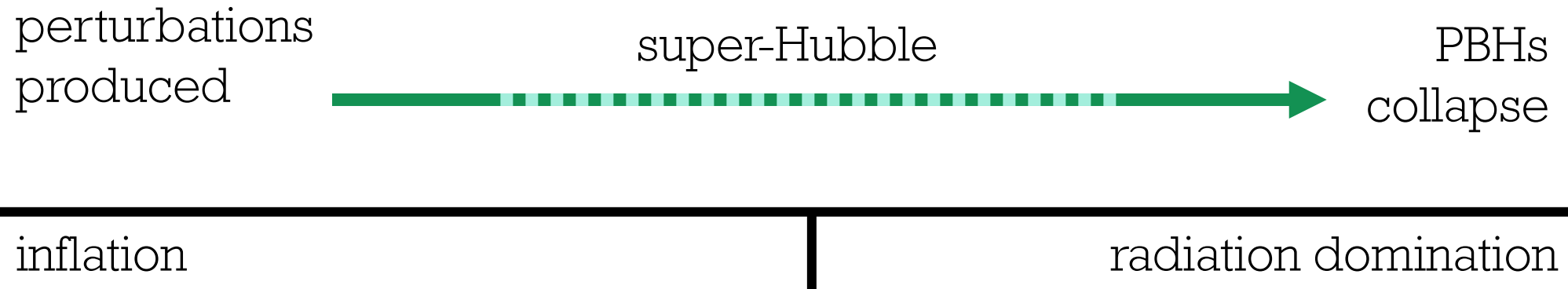


- Dark matter
- Seeds of supermassive BHs
- GW source

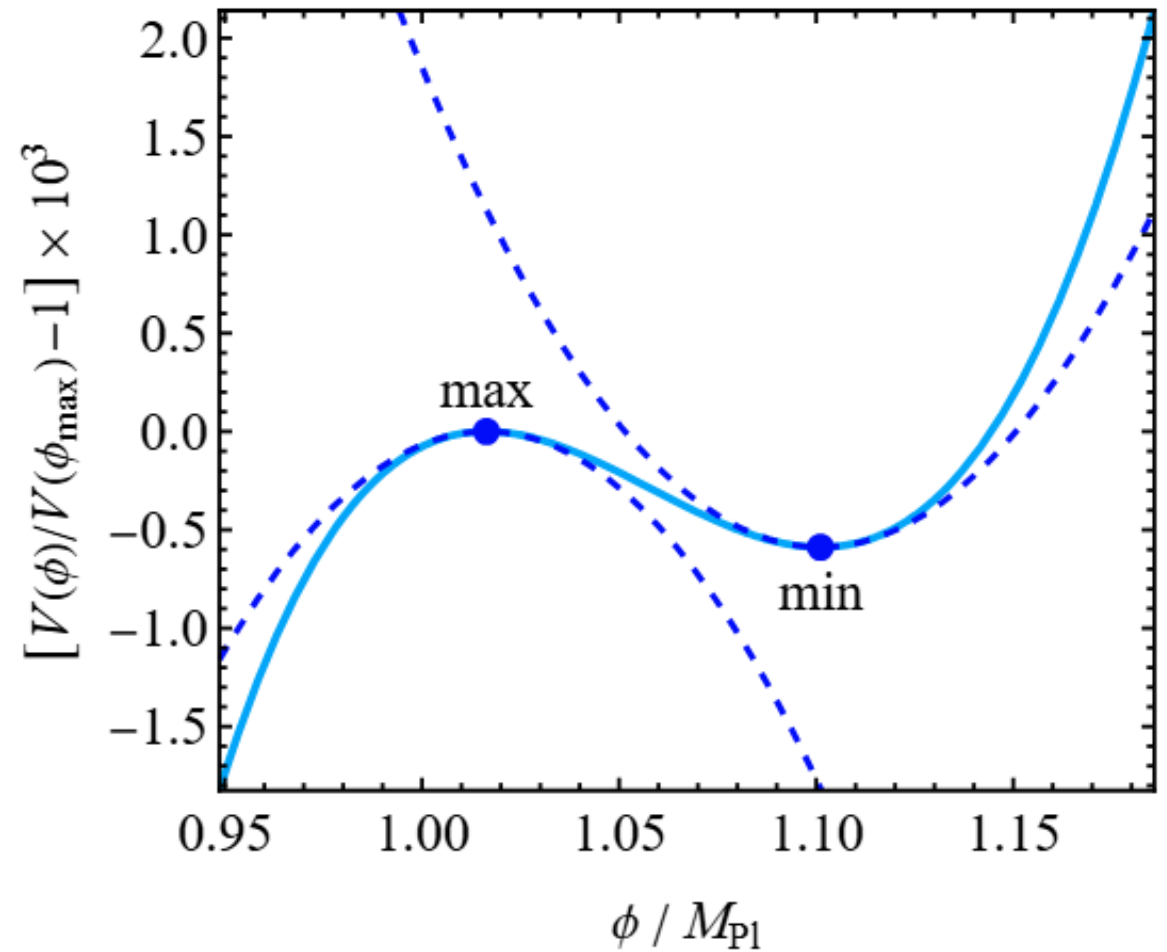
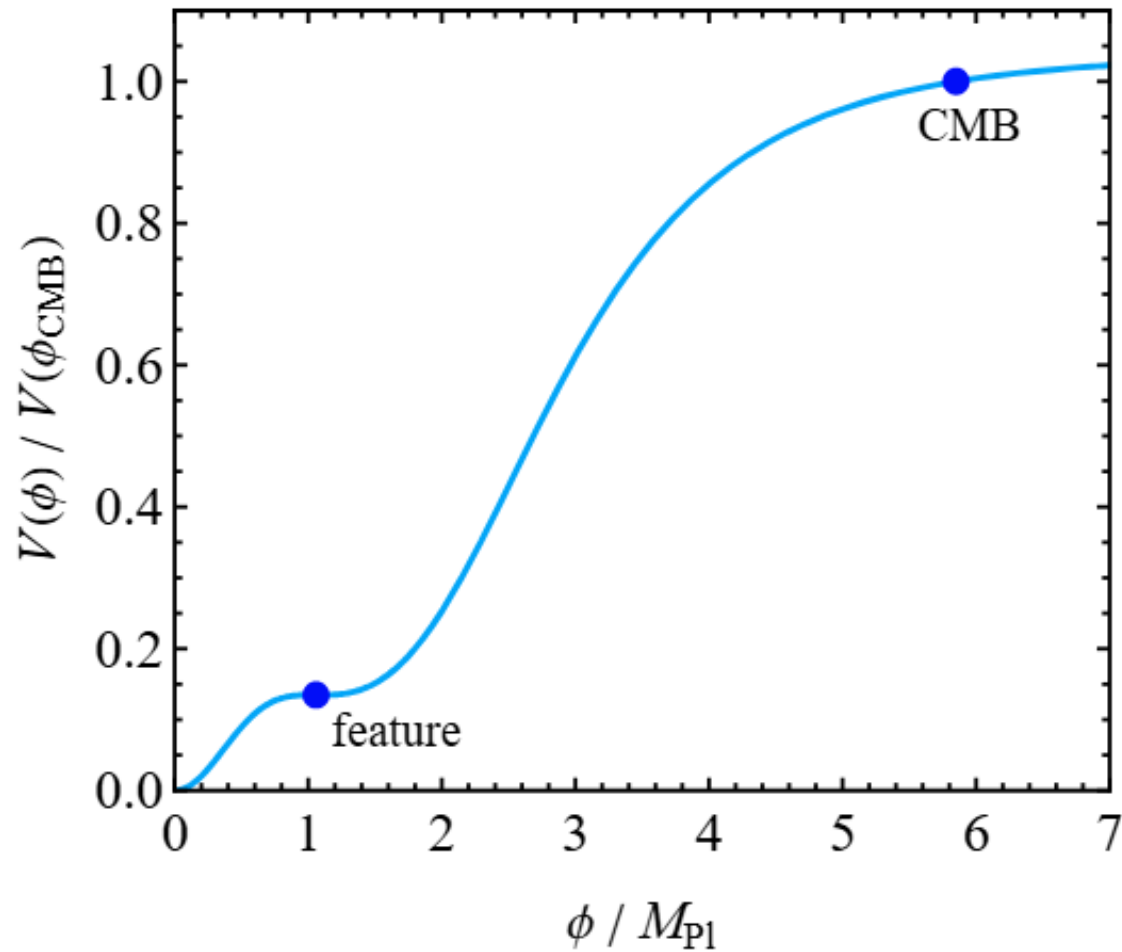
From compaction function:

$$\mathcal{C}(r) = 2 \frac{\delta M}{R} = \frac{2}{3} (1 - [1 + r\zeta'(r)]^2)$$

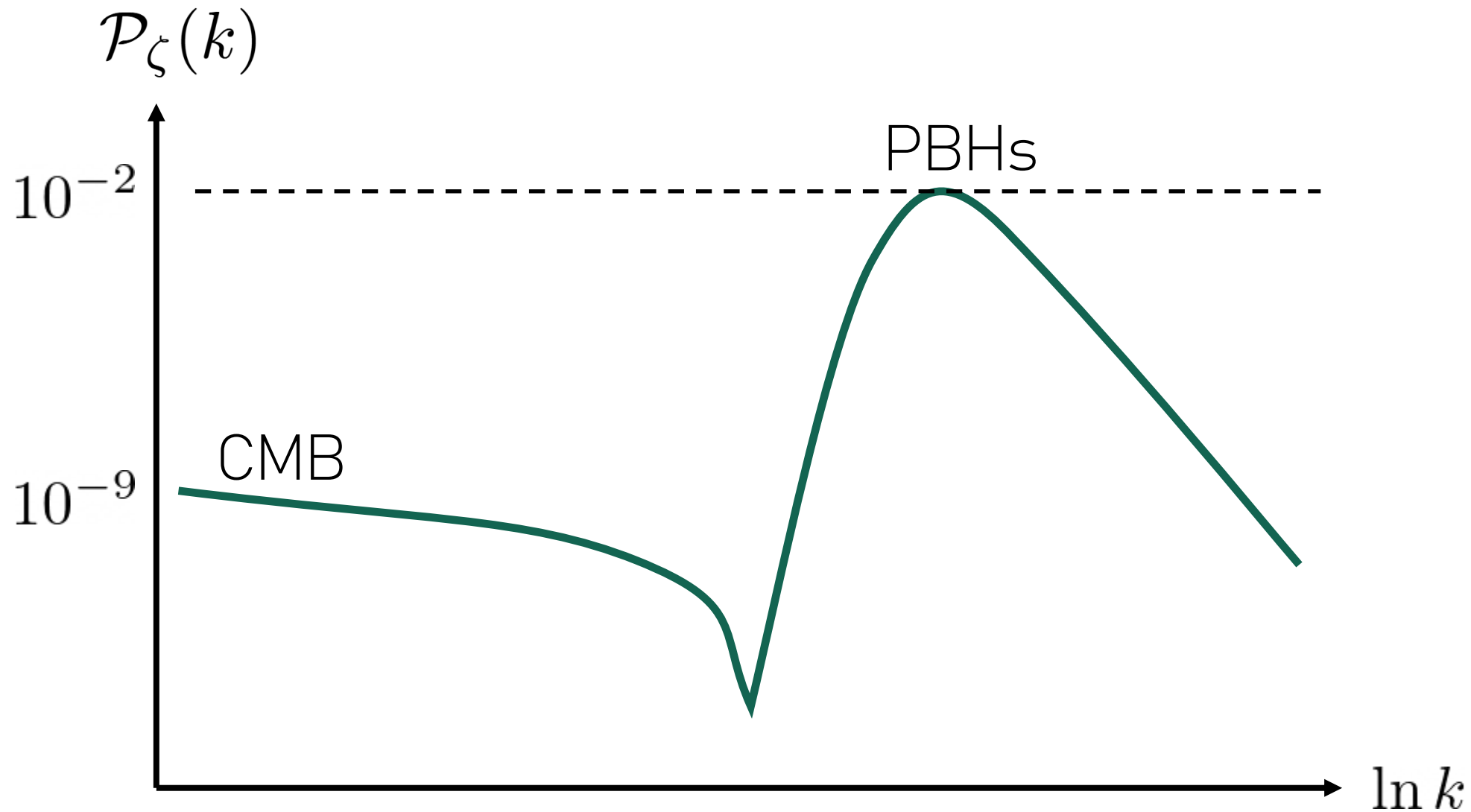
PBHs from inflation



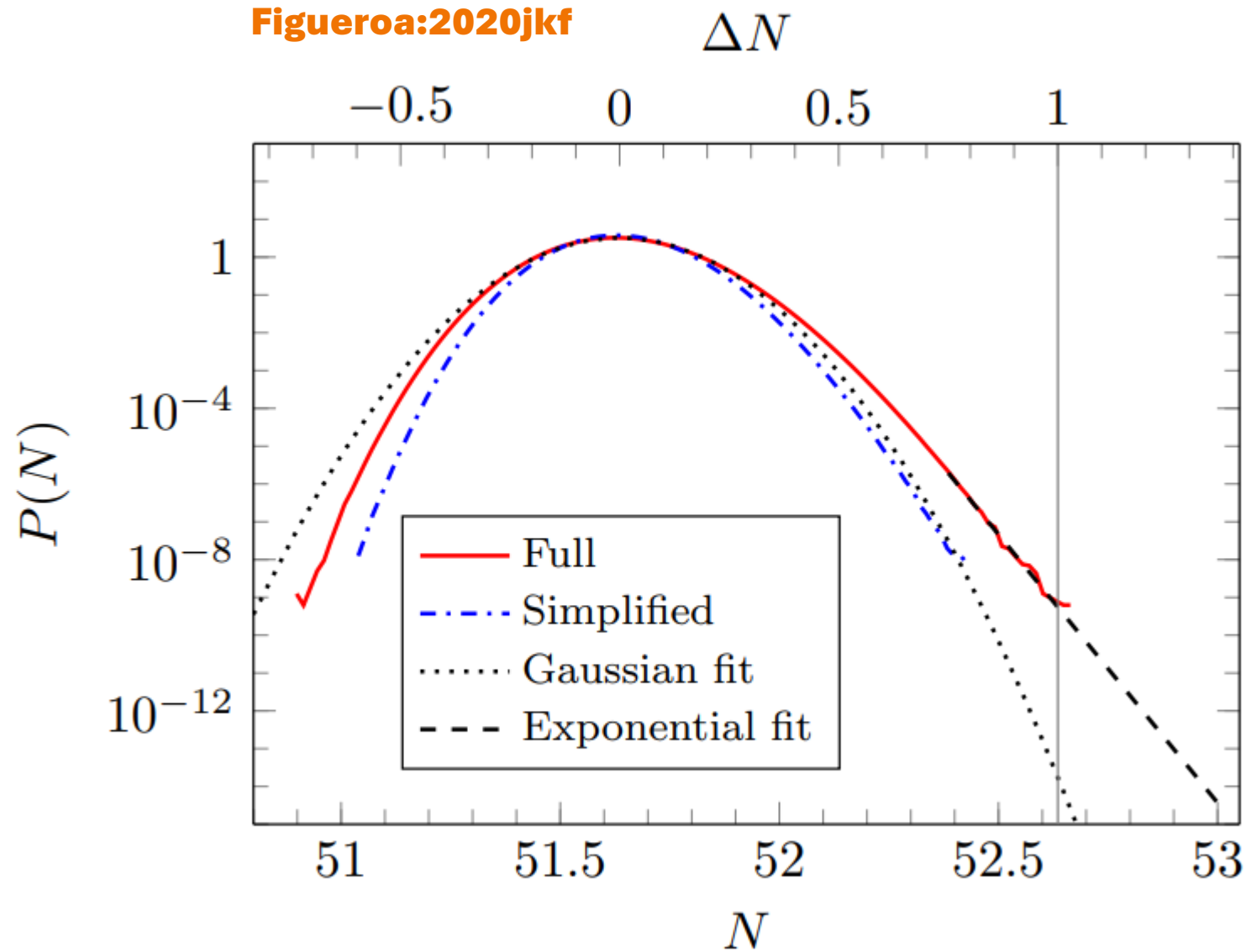
Inflection point models



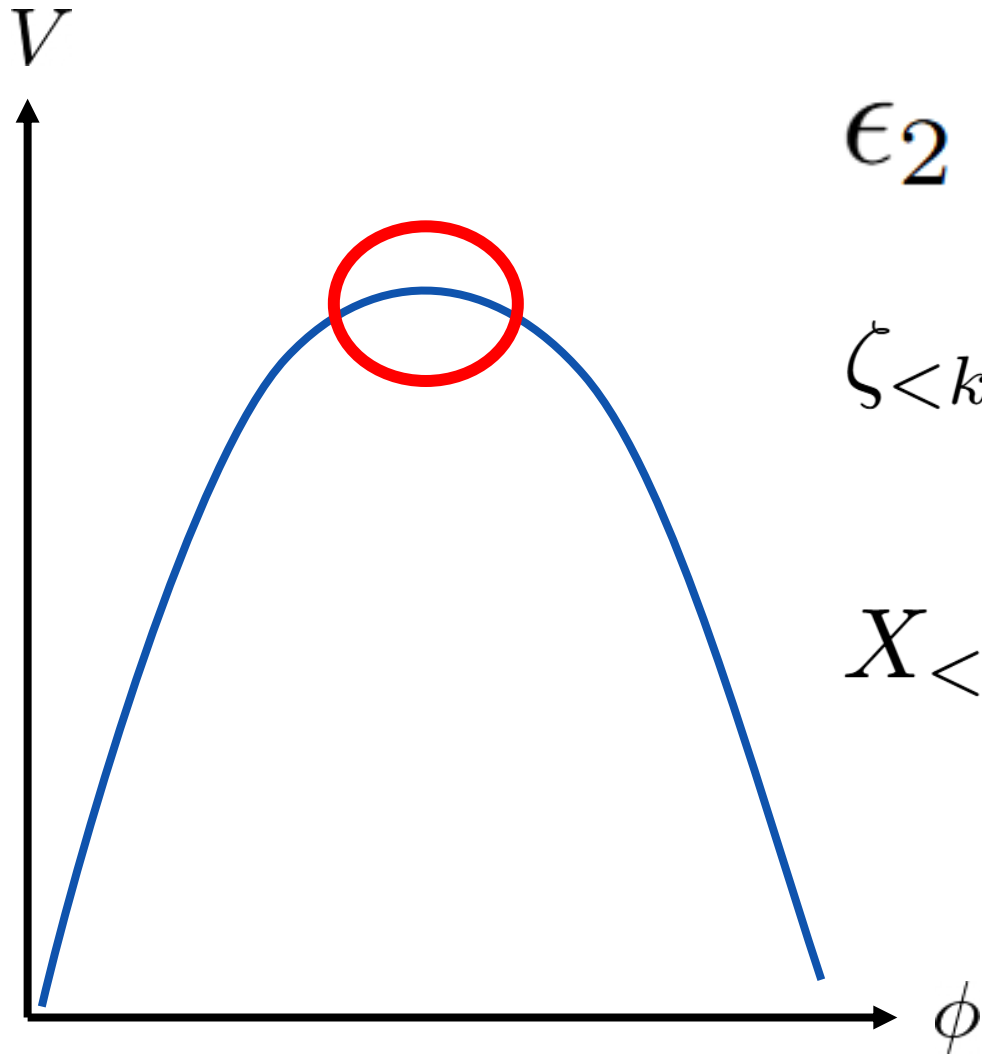
Inflection point models



Exponential tails



Simplification: constant roll



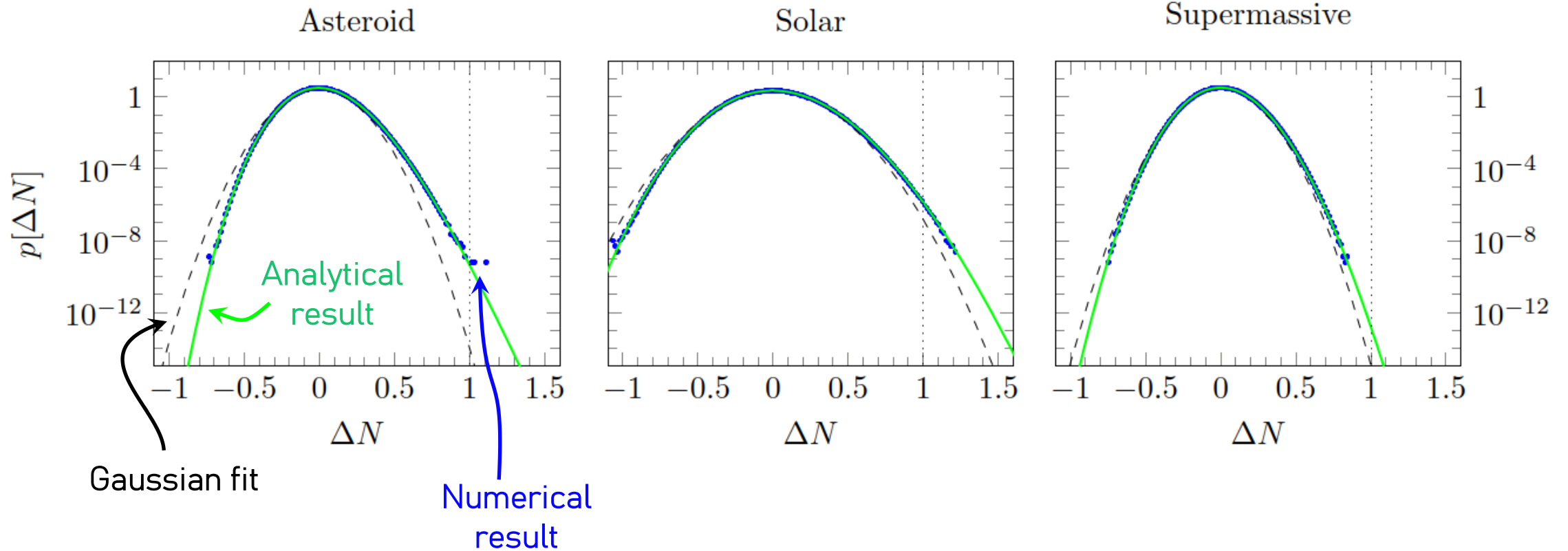
ϵ_2 approx. constant

$$\zeta_{<k} = \Delta N_{<k} = \frac{2}{\epsilon_2} \ln \left(1 - \frac{\epsilon_2}{2} X_{<k} \right)$$

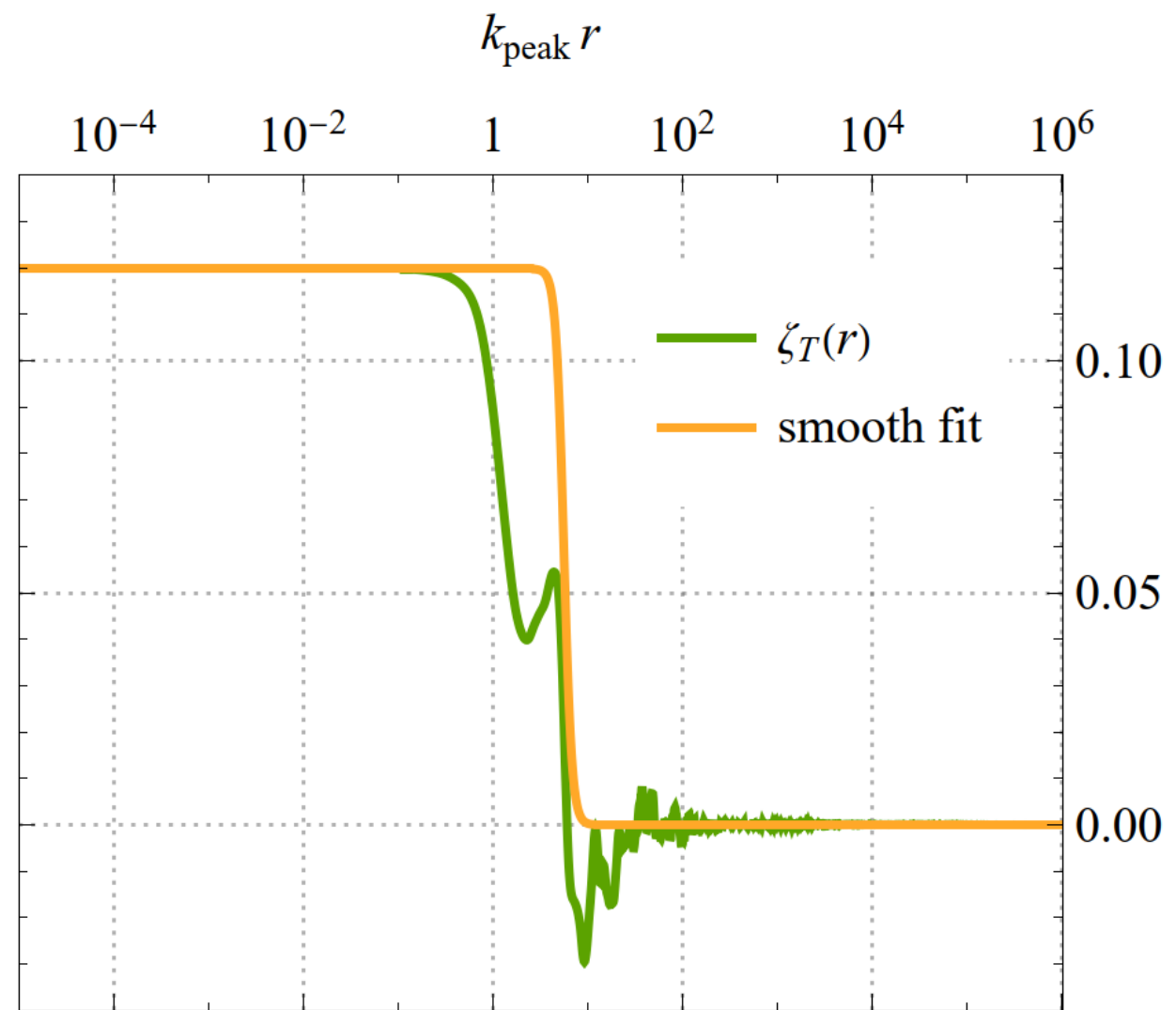
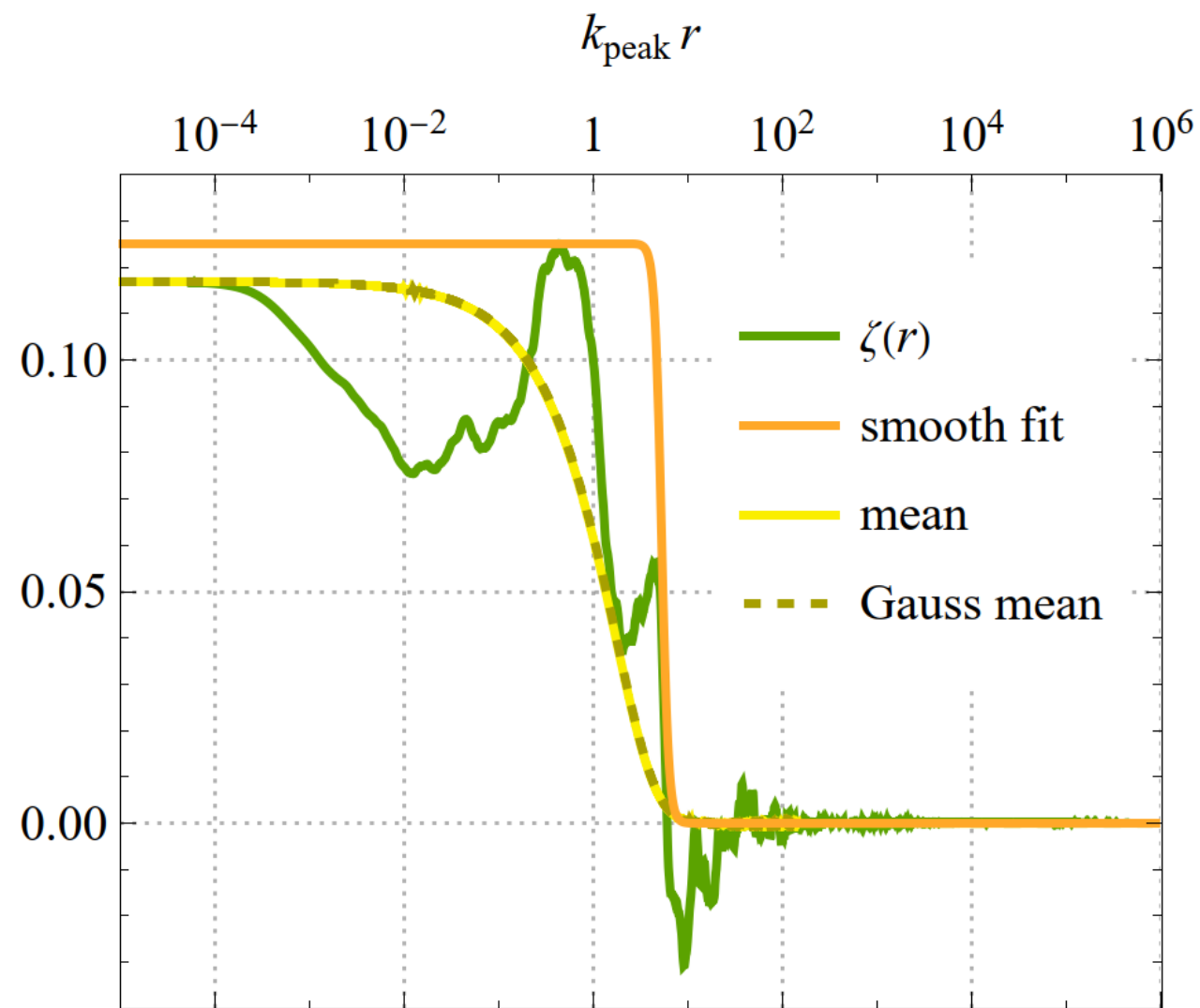
$$X_{<k} = - \int_{\tilde{k}=k_{\text{ini}}}^k \sqrt{\mathcal{P}_\zeta(\tilde{k})} d \ln k \hat{\xi}_{\tilde{k}}$$

Simplification: constant roll

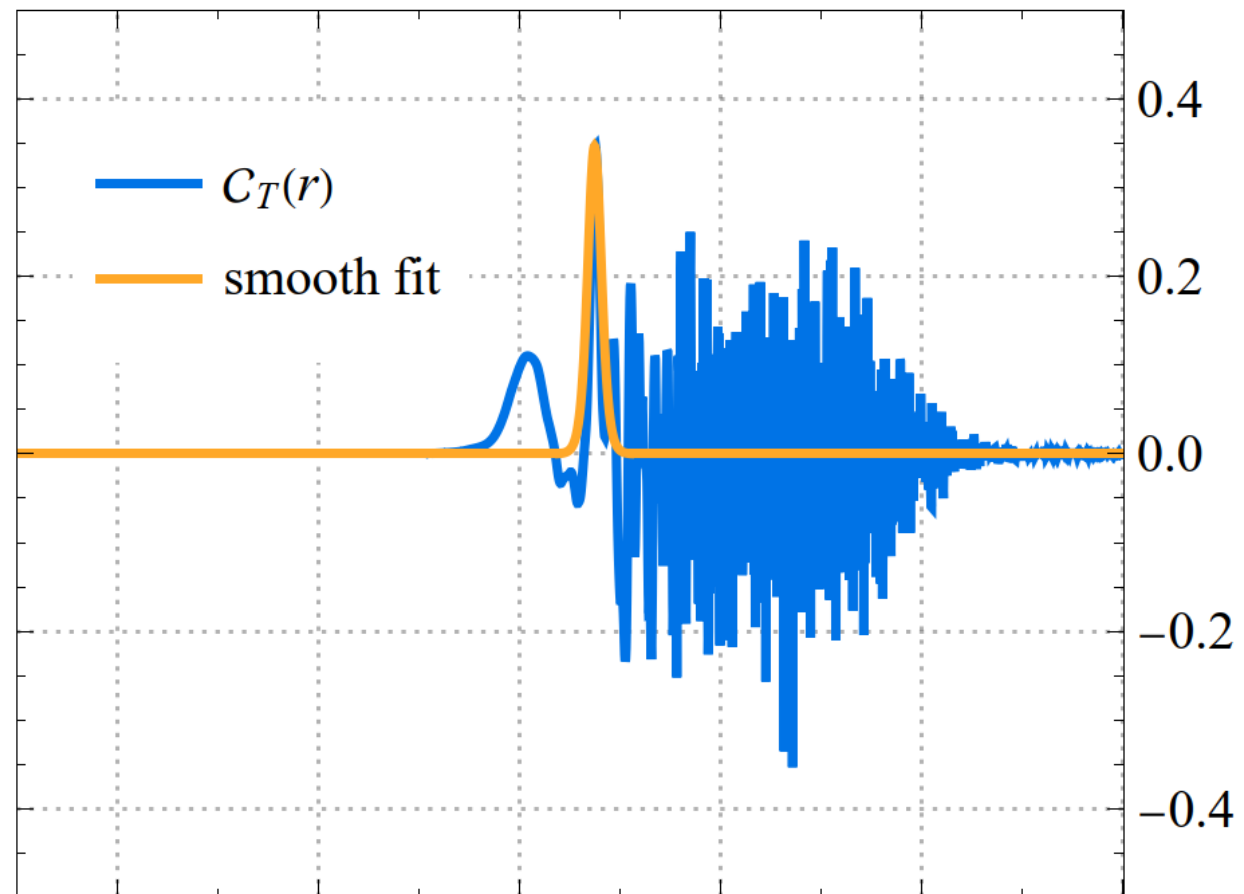
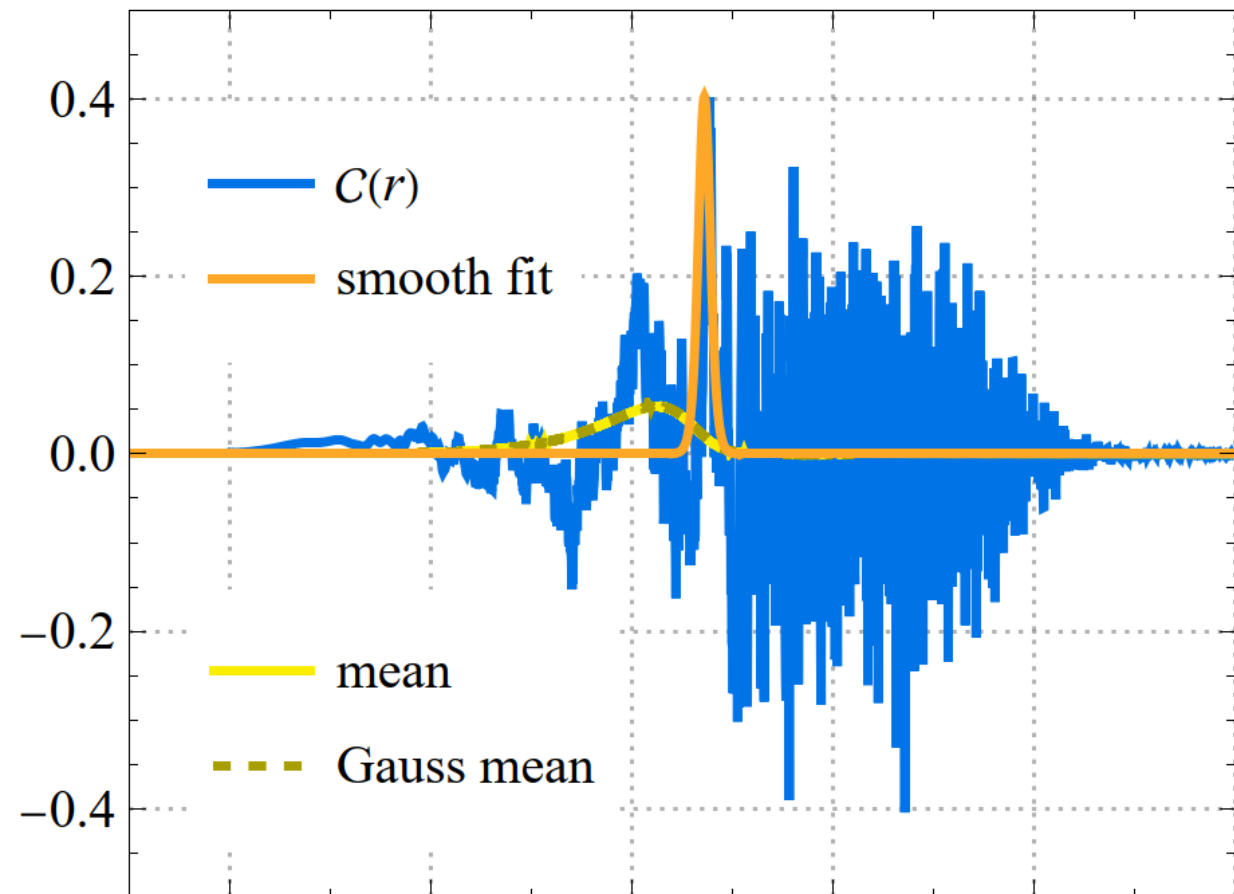
Tomberg:2023kli



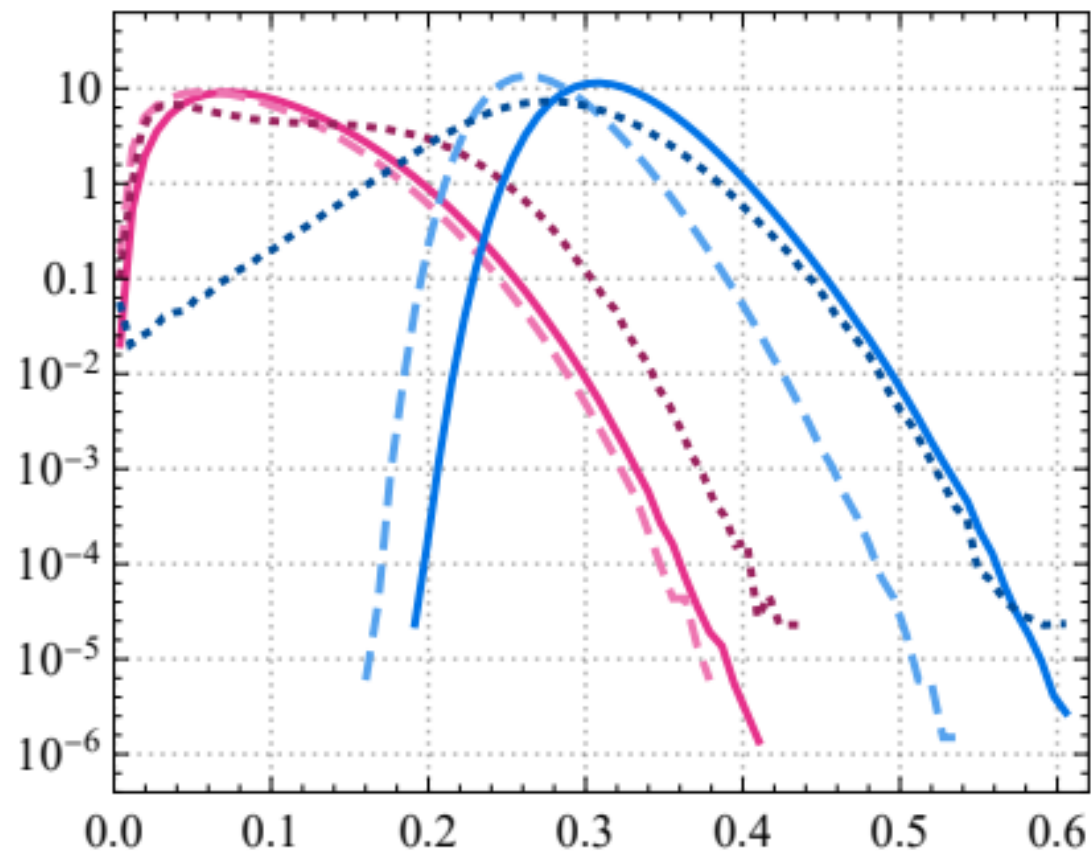
Raatikainen:2025gpd



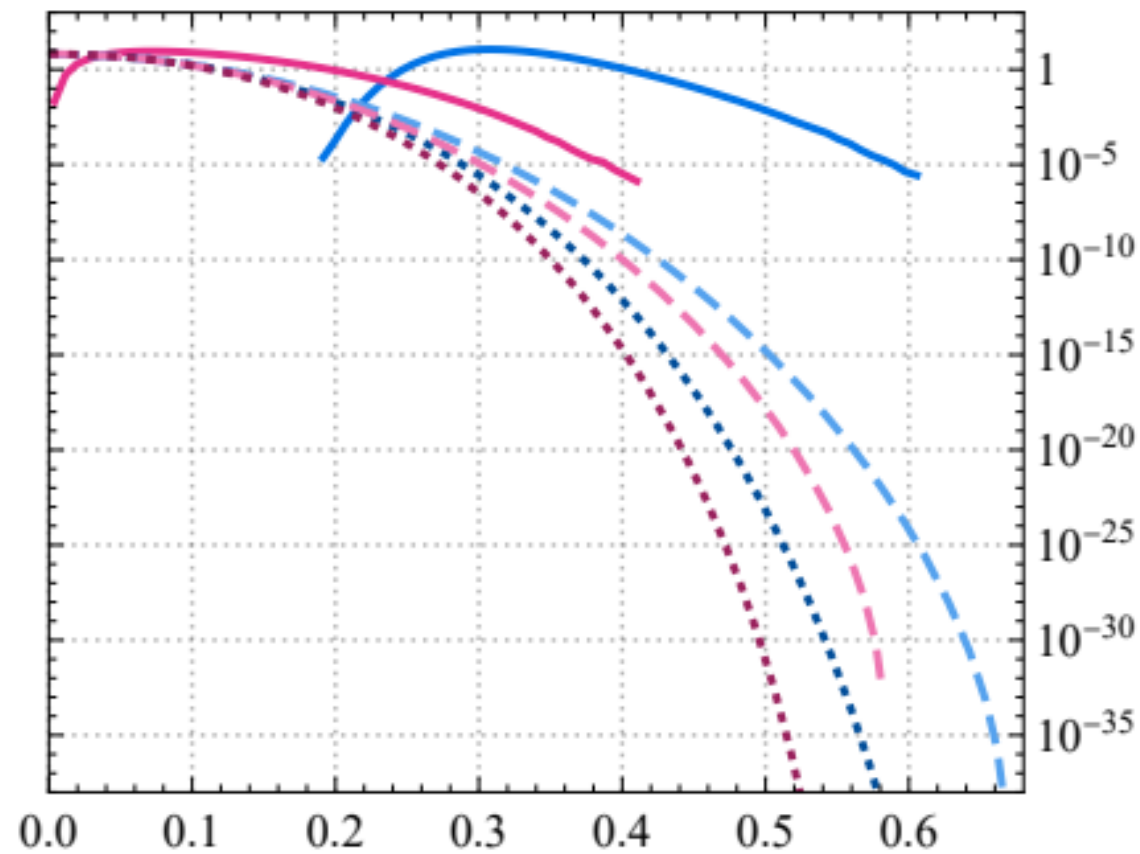
Raatikainen:2025gpd



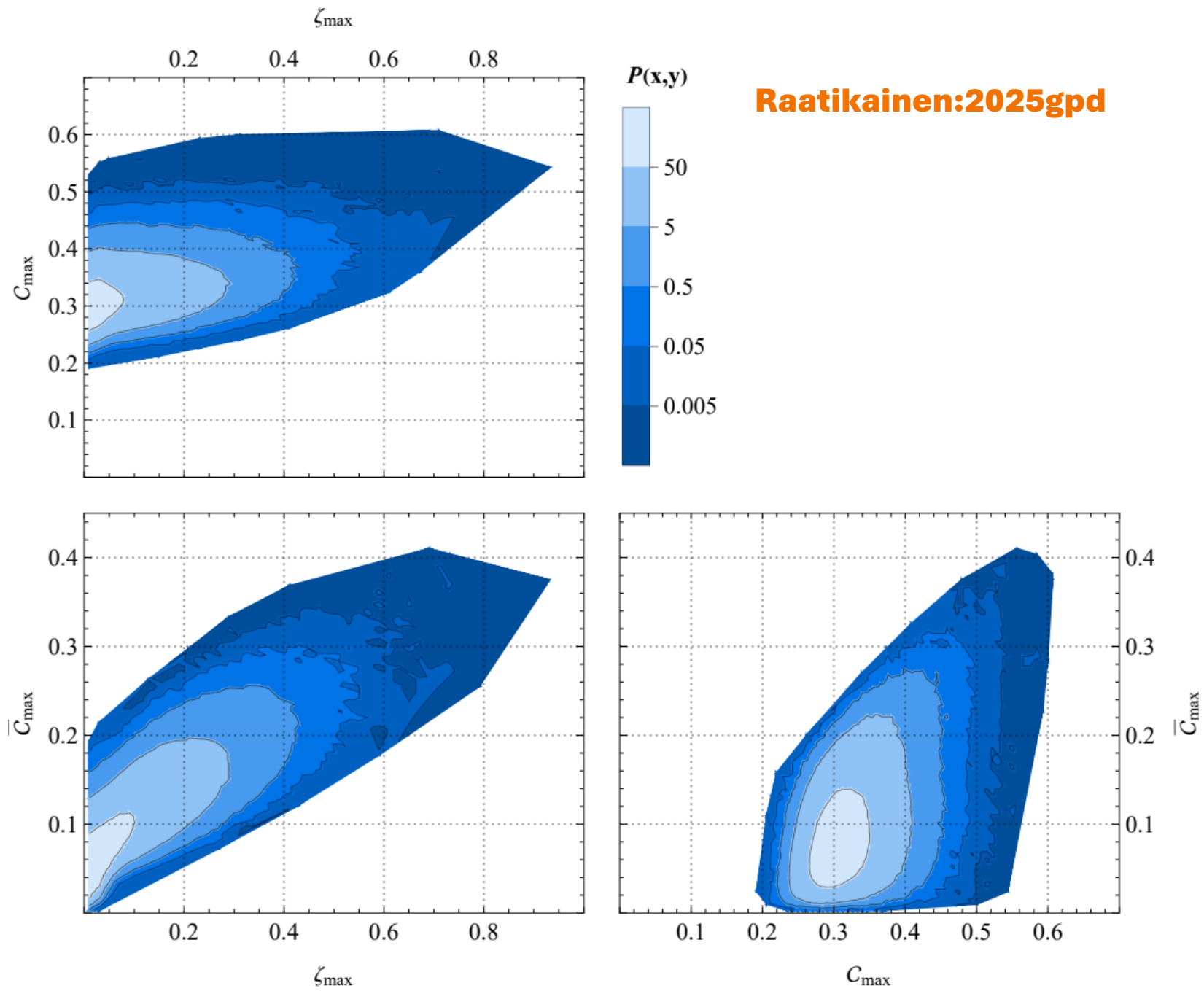
Raatikainen:2025gpd



— $P(C)$ - - - $P(C_T)$ ⋯ $P(C_{\text{smooth fit}})$
— $P(\bar{C})$ - - - $P(\bar{C}_T)$ ⋯ $P(\bar{C}_{\text{smooth fit}})$



— $P(C)$ - - - $P(C_{\text{mean}})$ ⋯ $P(C_{\text{Gauss mean}})$
— $P(\bar{C})$ - - - $P(\bar{C}_{\text{mean}})$ ⋯ $P(\bar{C}_{\text{Gauss mean}})$



Open questions

Convergence at large radius?

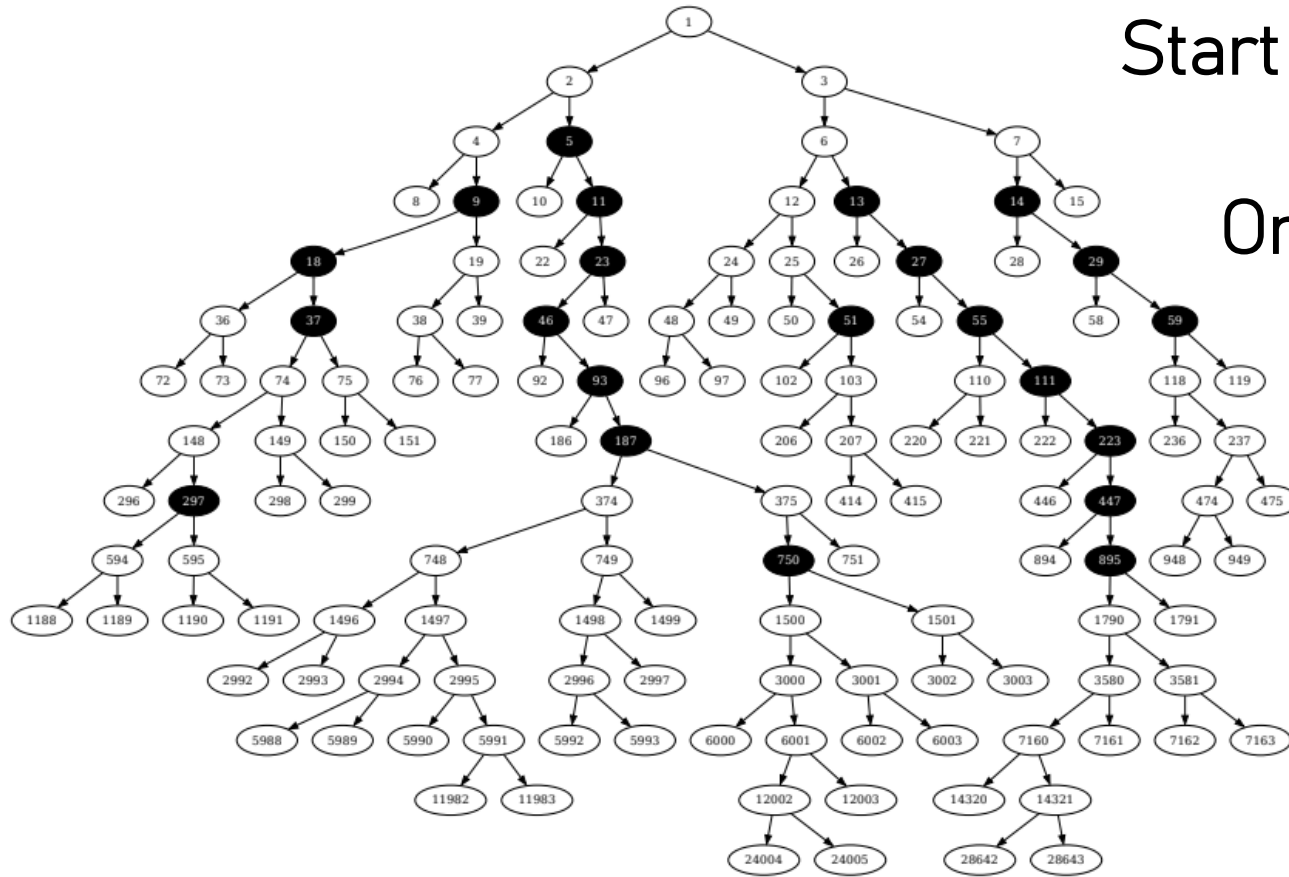
Correct collapse criterion for spiky profiles?

Redo collapse simulations?

Stochastic trees

Animali:2025pyf

Start from 1 Hubble patch



Once volume doubled: split in two

Spatial distribution?
Clustering?

Lattice simulations

Mizuguchi:2024kbl

Stochastic grid

Launay:2024qsm

Combining stochastics and numerical relativity

Stochastic Inflation in General Relativity

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¹*Centre for Theoretical Cosmology, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

²*School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne, NE1 7RU, United Kingdom*

We provide a formulation of Stochastic Inflation in full general relativity that goes beyond the slow-roll and separate universe approximations. We show how gauge invariant Langevin source terms can be obtained for the complete set of Einstein equations in their ADM formulation by providing a recipe for coarse-graining the spacetime in any small gauge. These stochastic source terms are defined in terms of the only dynamical scalar degree of freedom in single-field inflation and all depend simply on the first two time derivatives of the coarse-graining window function, on the gauge-invariant mode functions that satisfy the Mukhanov-Sasaki evolution equation, and on the slow-roll parameters. It is shown that this reasoning can also be applied to include gravitons as stochastic sources, thus enabling the study of all relevant degrees of freedom of general relativity for inflation. We validate the efficacy of these Langevin dynamics directly using an example in uniform field gauge, obtaining the stochastic e-fold number in the long wavelength limit without the need for a first-passage-time analysis. As well as investigating the most commonly used gauges in cosmological perturbation theory, we also derive stochastic source terms for the coarse-grained BSSN formulation of Einstein's equations, which enables a well-posed implementation for 3+1 numerical relativity simulations.

I. INTRODUCTION

Inflation theory was postulated more than 40 years ago as an explanation for the apparently fine-tuned initial conditions of the Hot Big Bang [1–3]. The proposal gained traction as it also offers a natural mechanism for generating the initial density inhomogeneities [4–9] which in later stages of cosmic history led to the formation of cosmic structure via gravitational instability. These den-

efforts [13–18], there are regimes where predictions can still be made by using techniques from Quantum Field Theory on Curved Spacetime (QFTCS) [15] or by constructing Effective Field Theories (EFTs), which have made continuous advancements in cosmology, inspired by the latter's success in flat space [19]. Abandoning pretenses of completeness, an EFT establishes a region of validity, normally bounded by ultraviolet (UV) and/or infrared (IR) cutoffs, and the narrative of theoretical physics is implicitly about pushing these cutoffs to their

RUP-24-10

PREPARED FOR SUBMISSION TO JCAP

STOLAS: STOchastic LAttice Simulation of cosmic inflation



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^cInstitute for Advanced Research, Nagoya University

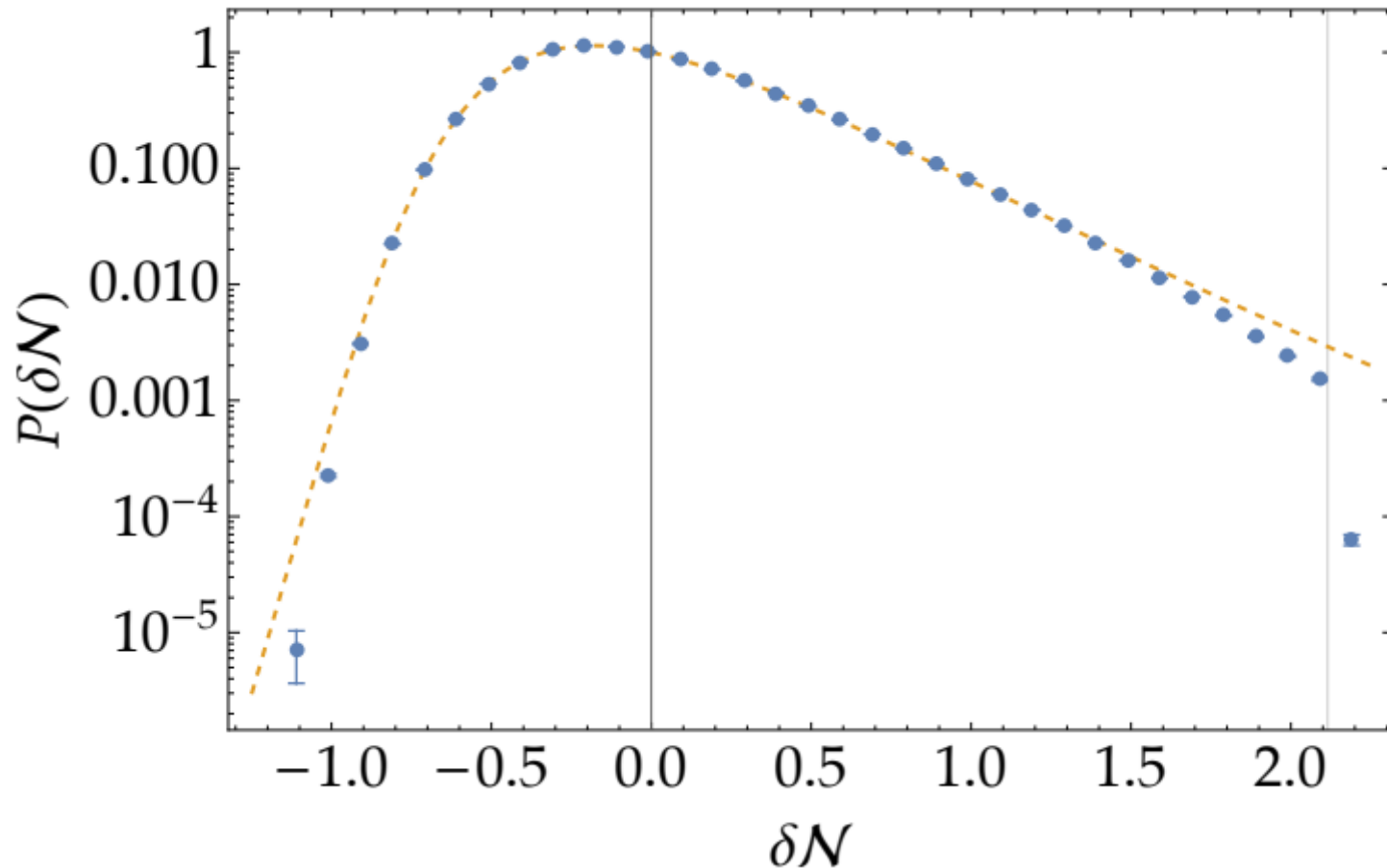
[astro-ph.CO] 20 Dec 2024

30v2 [gr-qc] 7 May 2024

Multi-field inflation

Murata:2025onc

Quadratic $n = 2$



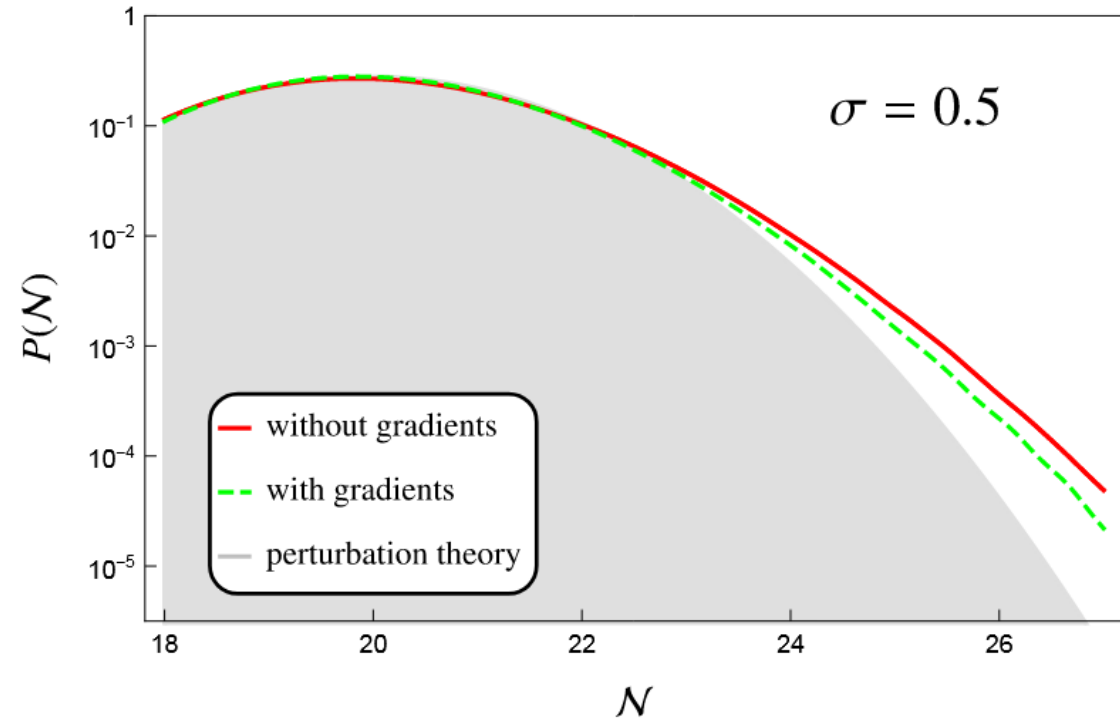
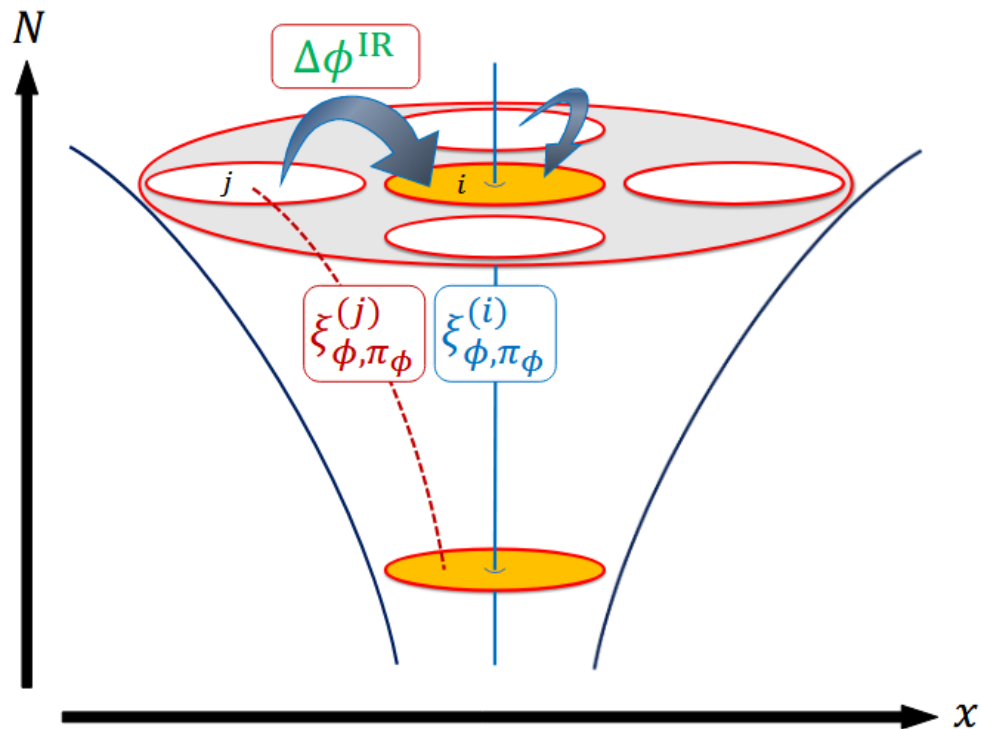
Hybrid inflation model

Upper bound to curvature perturbation

Gradient corrections

Briaud:2025ayt

Go beyond leading order in gradient expansion

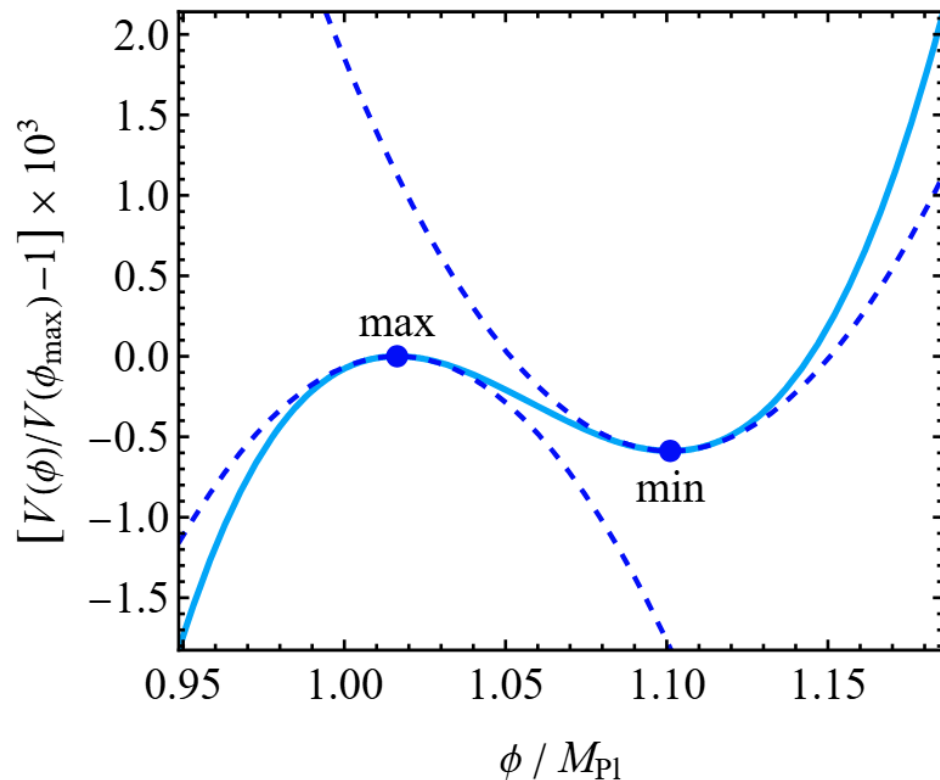


Colored noise
Exponential tails: a pullback effect

Eternal inflation

Tomberg:2025fku

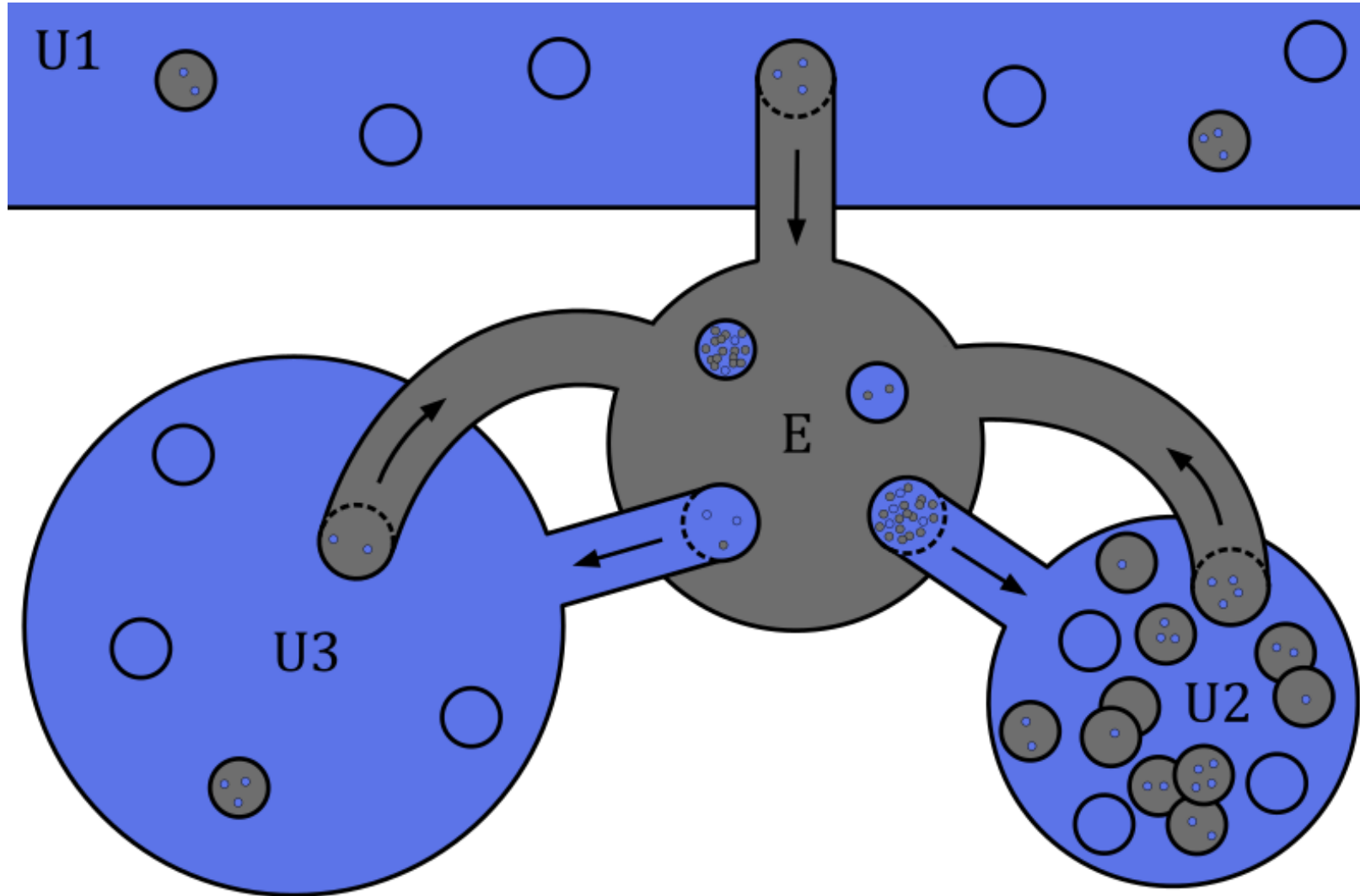
$$\text{eternal inflation} \iff \lim_{N \rightarrow \infty} \langle V \rangle_N \equiv \lim_{N \rightarrow \infty} \int_{\phi_{b1}}^{\phi_{b2}} e^{3N} P(\phi, N) d\phi > 0$$



$$P(\phi, N) \sim e^{-\lambda N} \quad \lambda \leq 3$$

$$\begin{array}{ll} \text{Maximum:} & \text{Minimum:} \\ \lambda \approx |\eta_H| \sim 0.1 & \lambda \approx 0 \end{array}$$

Eternal inflation



SUMMARY

Stochastic inflation gives a non-perturbative handle on inflationary perturbations

Useful for primordial black hole studies

Developments in various directions:
stochastic trees, multi-field setups,
gradient corrections...